Small Worlds Navigability: Lower Bound

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Joint work [ESA 2006] with:
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Interaction Networks

- Communication networks
  - Internet
  - Ad hoc and sensor networks
- Societal networks
  - The Web
  - P2P networks (the unstructured ones)
- Social network
  - Acquaintance
  - Mail exchanges
- Biology, linguistics, etc.

Common statistical properties

- Low density
- "Small world" properties:
  - Average distance between two nodes is small, typically $O(\log n)$
  - The probability p that two distinct neighbors u_1 and u_2 of a same node v are neighbors is large.

    p = clustering coefficient
- "Scale free" properties:
  - Heavy tailed probability distributions (e.g., of the degrees)

New problematic

- Why these networks share these properties?
- What model for
  - Performance analysis of these networks
  - Algorithm design for these networks
- Impact of the measures?
- This talk addresses **navigability**

Milgram Experiment

- Source person s (e.g., in Wichita)
- Target person t (e.g., in Cambridge)
  - Name, professional occupation, city of living, etc.
- Letter transmitted via a chain of individuals related on a **personal** basis
- Result: “six degrees of separation”

NAVIGABILITY

- Jon Kleinberg (2000)
  - Why should there **exist** short chains of acquaintances linking together arbitrary pairs of strangers?
  - Why should arbitrary pairs of strangers be able to **find** short chains of acquaintances that link them together?
- In other words: how to **navigate** in a small worlds?

Augmented graphs H=G+D

- Individuals as nodes of a graph G
  - Edges of G model relations between individuals deducible from their societal positions
- A "**Long link**" is added to every node of G at random, according to probability distribution D
  - Long links model relations between individuals that **cannot** be deduced from their societal positions

Greedy Routing in augmented graphs

- Source $s \in V(G)$
- Target $t \in V(G)$
- Current node x selects among its $deg_G(x)+1$ neighbors the closest to t in G, that is according to the distance function $dist_G()$.

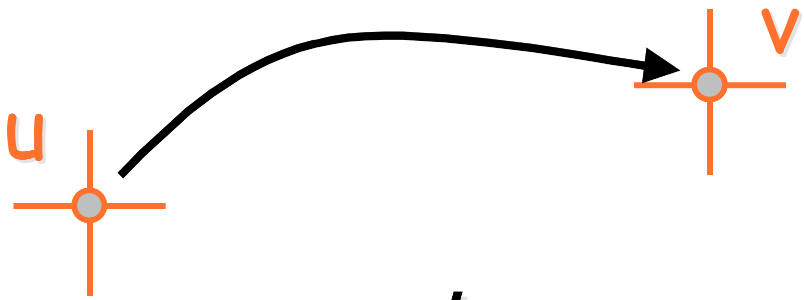
Greedy routing in augmented graphs aims at modeling the routing process performed by social entities in Milgram's experiment.

Augmented meshes

Kleinberg [STOC 2000]

d-dimensional n-node meshes

augmented with d-harmonic links



$$\text{prob}(u \rightarrow v) \approx 1/(\log(n)^{*}\text{dist}(u,v)^{d})$$

Kleinberg's theorems

- Greedy routing performs in $O(\log^2 n)$ expected #steps in augmented

  d-dimensional meshes whose long links are chosen according to the d-harmonic distribution.

Extensions

- Two-step greedy routing: $O(\log n / \log\log n)$
  - Coppersmith, Gamarnik, Sviridenko (2002)
    - Percolation theory
  - Manku, Naor, Wieder (2004)
    - NoN routing
- Routing with partial knowledge: $O(\log^{1+1/d} n)$
  - Martel, Nguyen (2004)
    - Non-oblivious routing
  - Fraigniaud, Gavoille, Paul (2004)
    - Oblivious routing
- Decentralized routing: $O(\log n * \log^2 \log n)$
  - Lebhar, Schabanel (2004)
    - $O(\log^2 n)$ expected #steps to find the route

Navigable graphs

- Let f : **N** $\rightarrow$ **R** be a function
- An n-node graph G is f-navigable if there exists an augmentation D for G such that greedy routing in G+D performs in at most f(n) expected #steps.
- I.e., for any two nodes u,v we have

$$\mathbf{E}_D(\#steps_{u\rightarrow v}) \leq f(n)$$

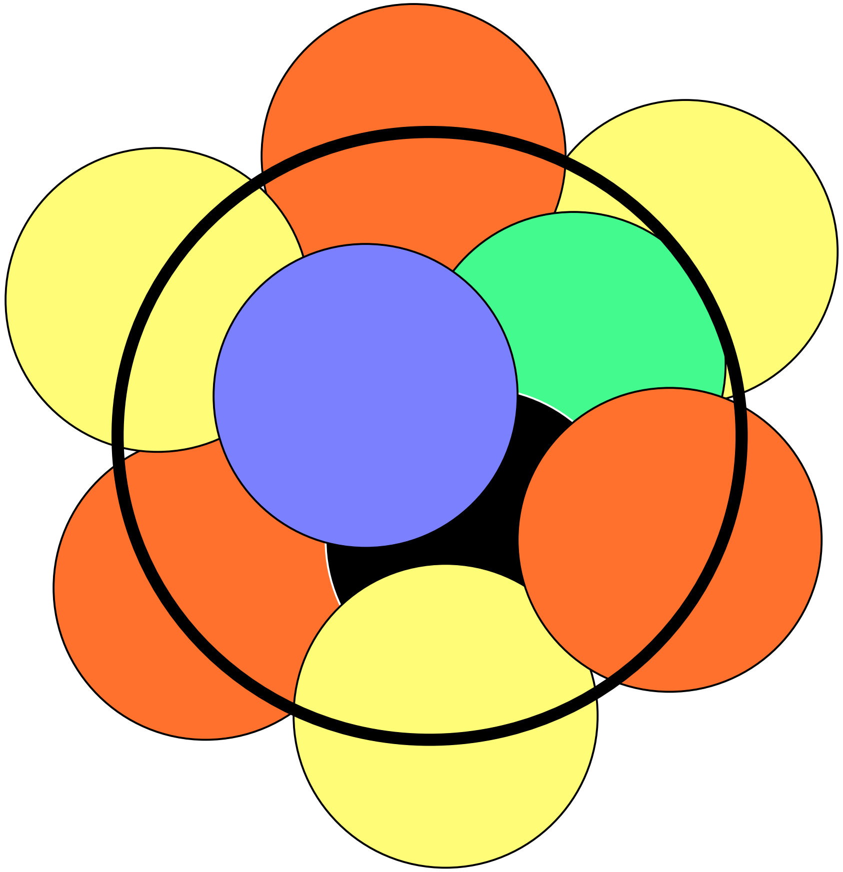
POLYLOG(N)-NAVIGABLE GRAPHS

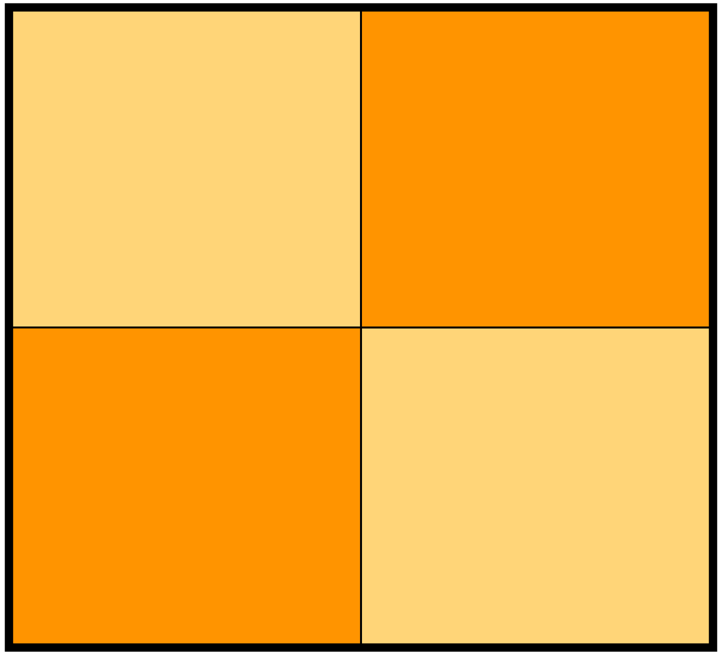
- Bounded growth graphs
  - Definition: $|B(x,2r)| \leq \rho\, |B(x,r)|$
  - Duchon, Hanusse, Lebhar, Schabanel (2005,2006)
- Bounded doubling dimension
  - Definition: Every $B(x,2r)$ can be covered by at most 2^d balls of radius r
  - Slivkins (2005)
- Graphs of bounded treewidth
  - Fraigniaud (2005)
- Graphs excluding a fixed minor
  - Abraham, Gavoille (2006)

Question

Are all graphs polylog(n)-navigable?

Doubling dimension





Svilkins' theorem

Any family of graphs with doubling dimension O(loglog n) is polylog(n)-navigable.

Impossibility result

<u>Theorem</u>

Let d such that

$$\lim_{n \to +\infty} \log\log n \,/\, d(n) = 0$$

There exists an infinite family of n-node graphs with doubling dimension at most d(n) that are not polylog(n)-navigable.

<u>Consequences</u>:

1. Svilkins' result is tight
2. Not all graphs are polylog(n)-navigable

PROOF OF NON-NAVIGABILITY

The graphs H_d with $n=p^d$ nodes

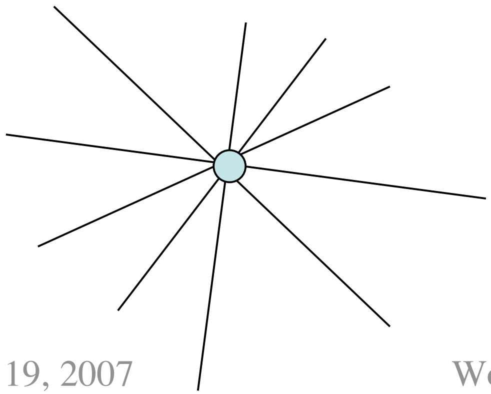
$x = x_1\, x_2 \ldots x_d$
is connected to all nodes
$y = y_1\, y_2 \ldots y_d$
such that $y_i = x_i + a_i$ where
$a_i \in \{-1,0,+1\}$

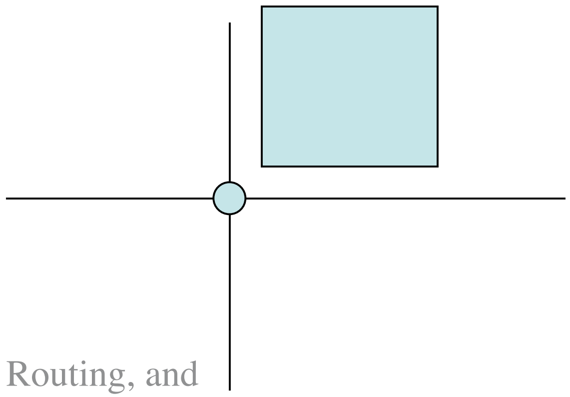
H_d has doubling dimension d

Intuitive approach

- Large doubling dimension d

  $\Rightarrow$ every nodes $x \in H_d$ has choices over exponentially many directions
- The underlying metric of H_d is L_∞

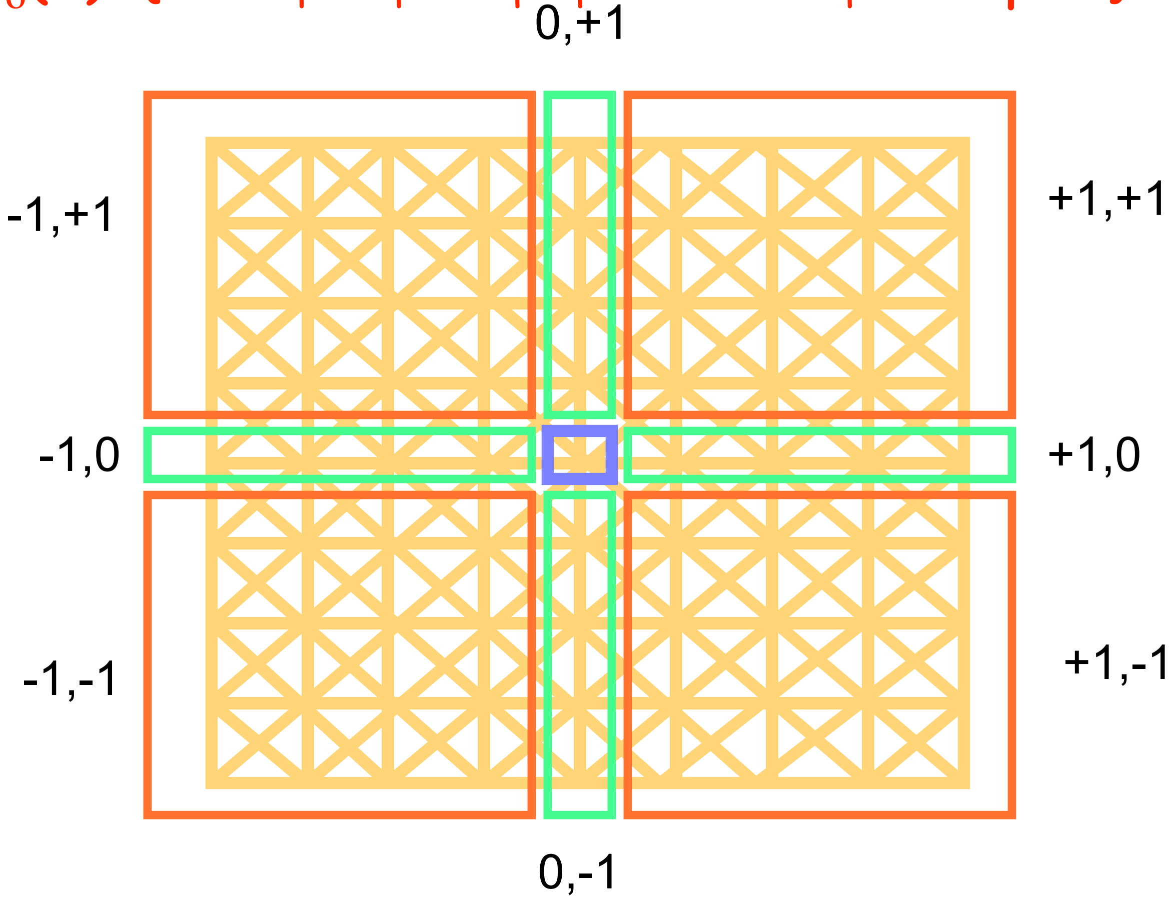




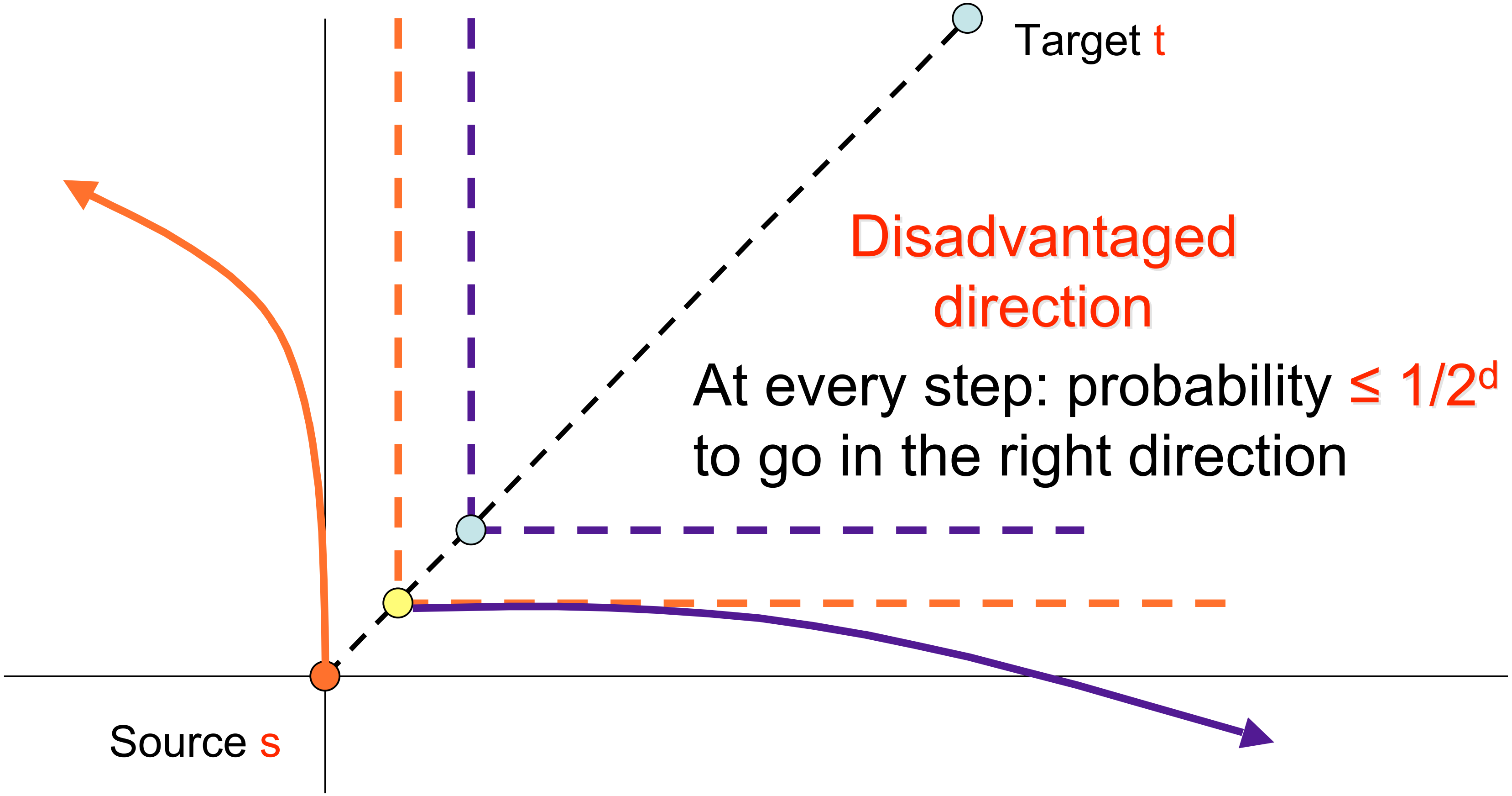
Directions

$\delta = (\delta_1, \ldots, \delta_d)$ where $\delta_i \in \{-1,0,+1\}$

$Dir_\delta(u)=\{v \;/\; v_i = u_i + x_i\,\delta_i$ where $x_i = 1\ldots p/2\}$

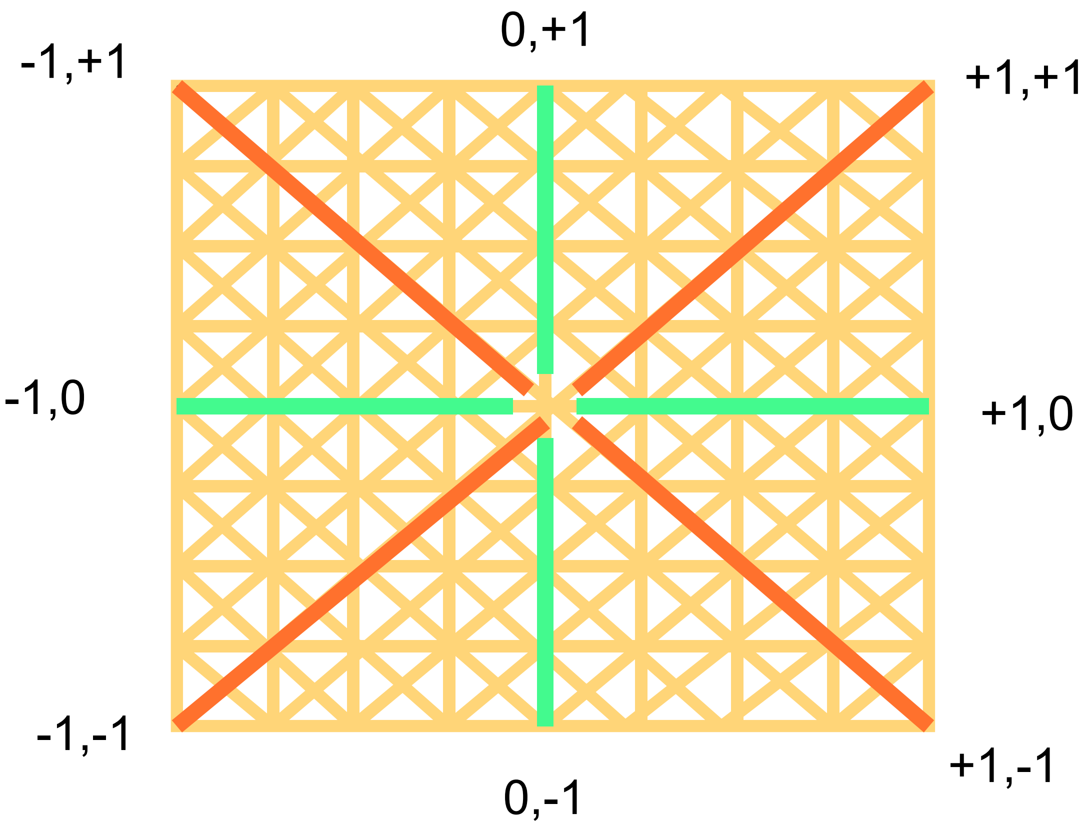


Case of symmetric distribution

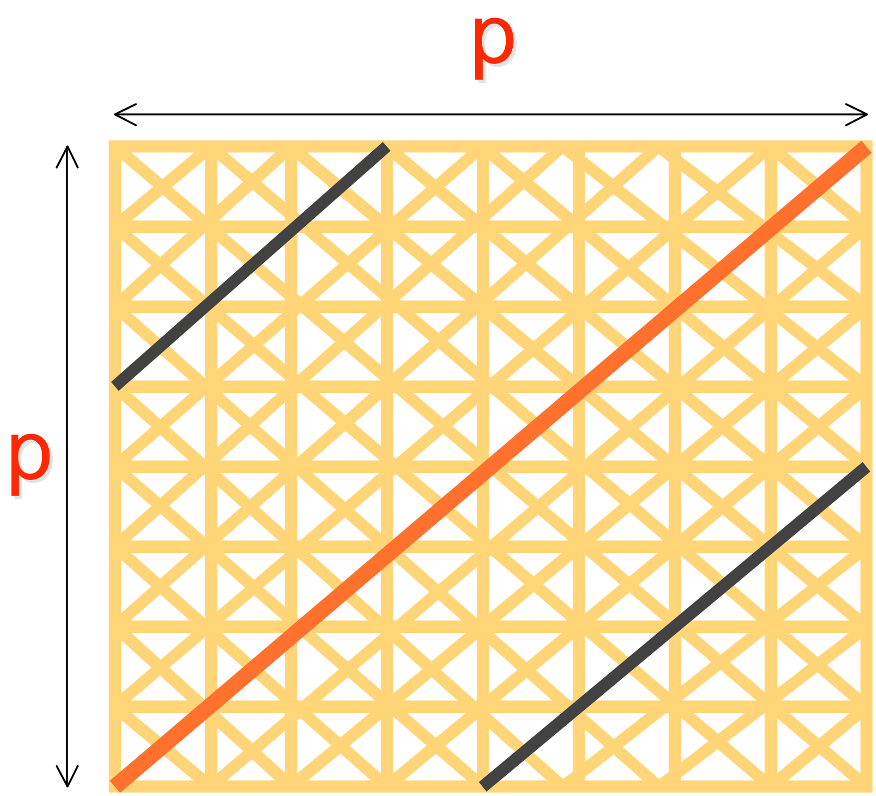


-- General case --

Diagonals

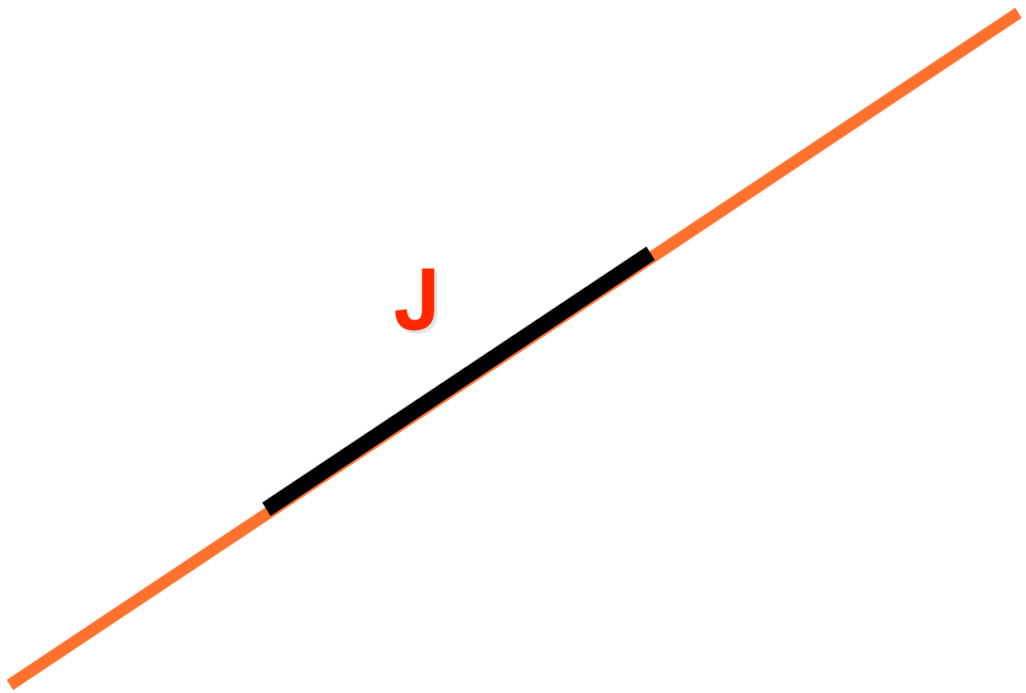


Lines

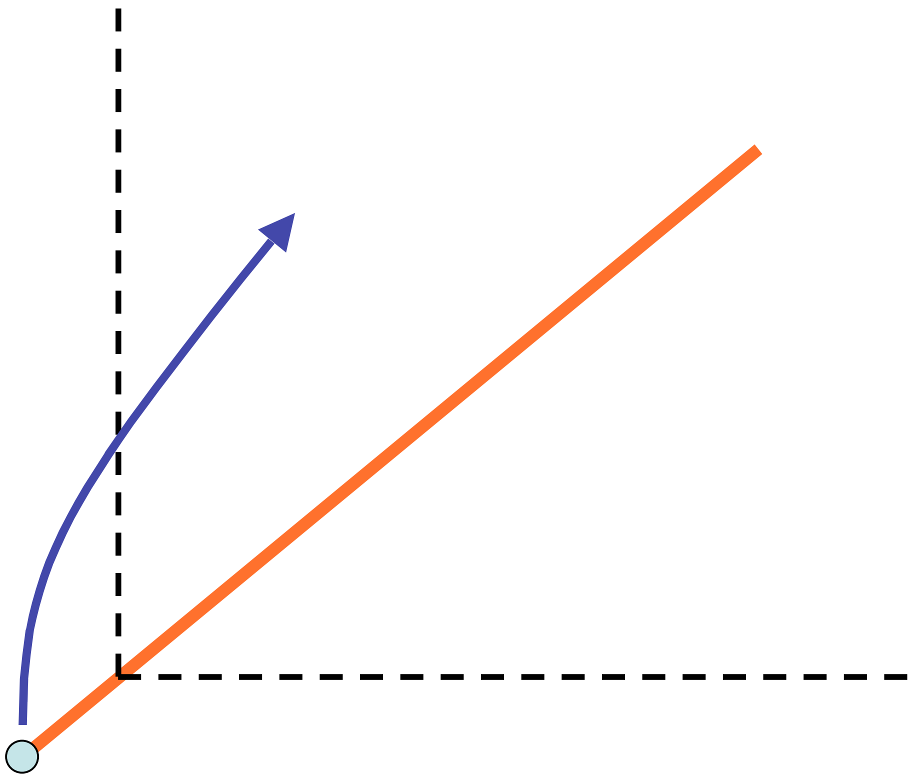


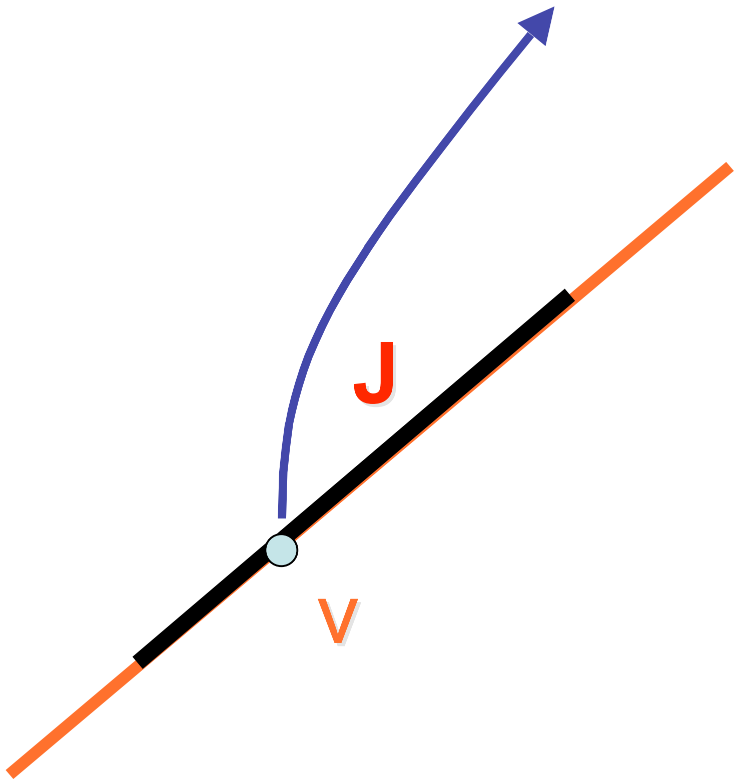
p lines in each direction

Intervals



Certificates



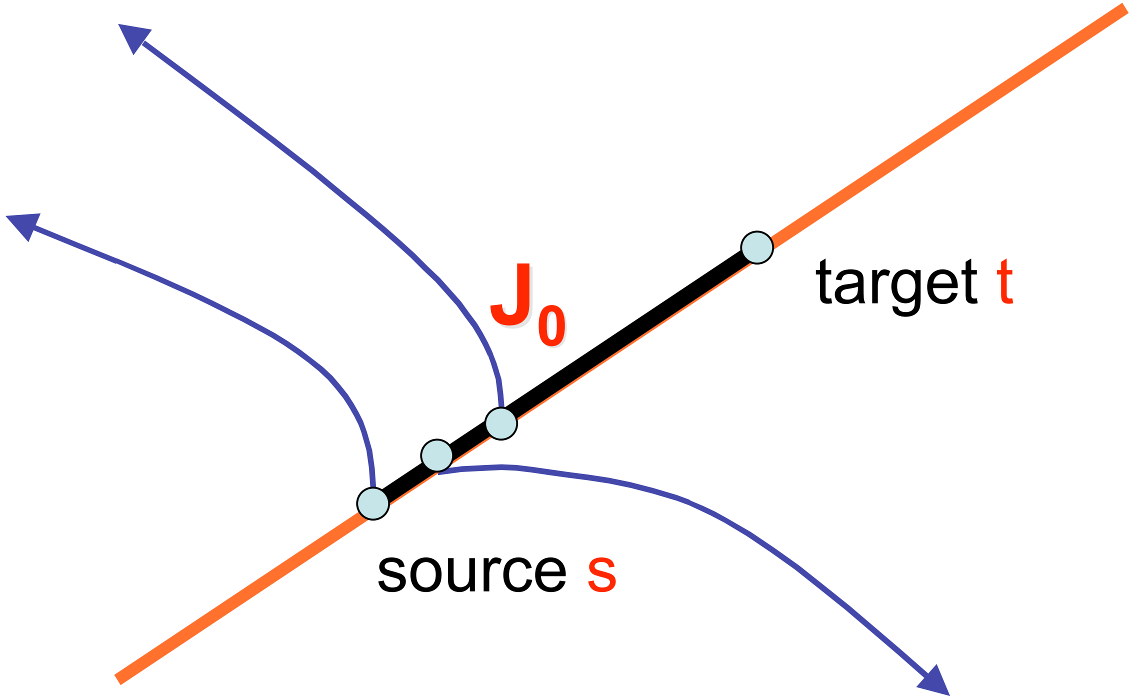


v is a certificate for J

Counting argument

- 2^d directions
- Lines are split in intervals of length L
- $n/L \times 2^d$ intervals in total
- Every node belongs to many intervals, but can be the certificate of at most one interval
- If $L<2^d$ (say $L= 2^d -1$) then there is one interval J_0 without certificate

L-1 steps from s to t



In expectation…

- $n/L \times 2^d - n$ intervals without certificate
- $L = 2^{d-1} \Rightarrow n$ of the $2n$ intervals are without certificate
- This is true for any trial of the long links
- Hence $E = \mathbf{E}_D$(#interval without certificate) $\geq n$
- On the other hand:

$$E = \sum_J \Pr(J \text{ has no certificate})$$

- Hence there is an interval $J_0=[s,t]$ such that

$$\Pr(J_0 \text{ has no certificate}) \geq 1/2$$

- Hence $\mathbf{E}_D(\#steps_{s\rightarrow t}) \geq (L-1)/2$ **QED**

Remark: This holds even if the long links are not set pairwise independently.

Corollaries

- No universal augmentation scheme can yield the expected number of steps of greedy routing below $\Omega(n^{1/\sqrt{\log(n)}})$
- Indeed: We set $L \approx 2^d$, but under the constraint $L \leq n^{1/d}$

Open Problem

- Close the gap between $n^{1/3}$ and $n^{1/\sqrt{\log(n)}}$
- We have considered the worst case:

$$\max_{u,v} \mathbf{E}_D(\#steps_{u \to v})$$

What about the average case?

$$\Sigma_{u,v} \mathbf{E}_D(\#steps_{u \to v}) / n^2$$