Navigability in small worlds: upper bounds

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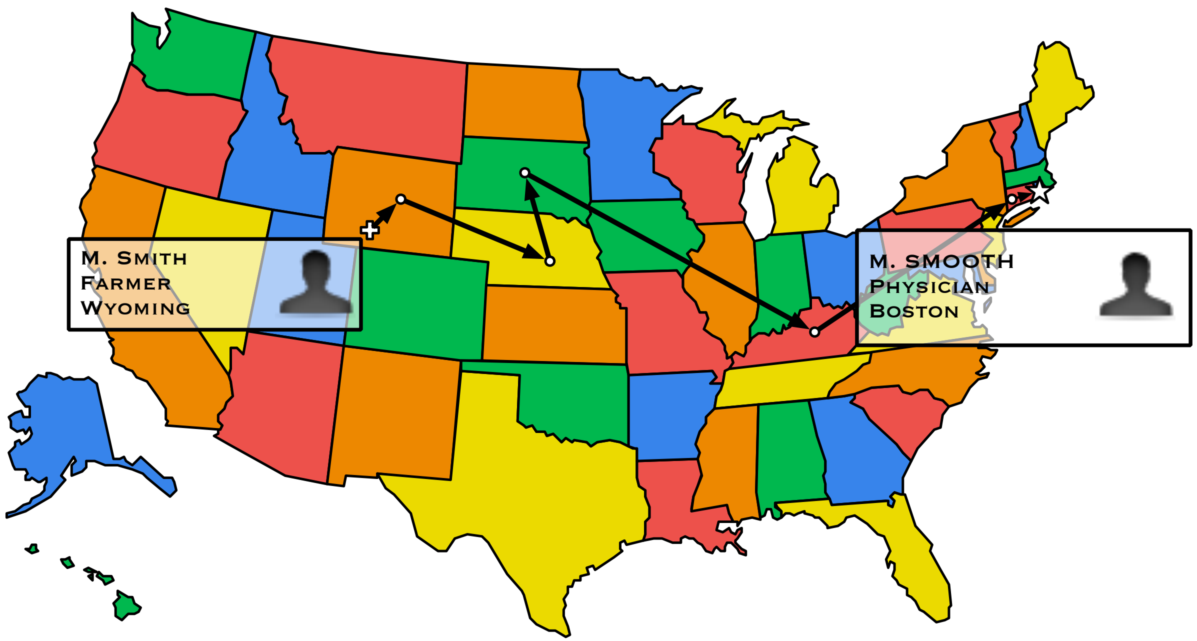
Navigability in small worlds

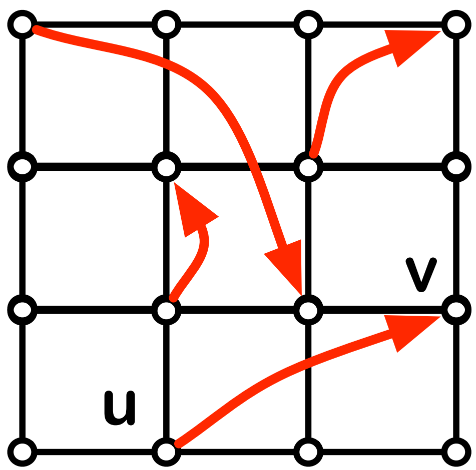
- "Small world effect": discovering very short paths with only a local view.



- Model [Kleinberg'00]:
  - Greedy routing for decentralized search.
  - Augmented graphs for network structure.

Kleinberg model

- Base graph globally known: regular 2D mesh.
- + Additional links locally known: randomly distributed along $Pr\{u \rightarrow v\}$ prop to $1/|u-v|^2$



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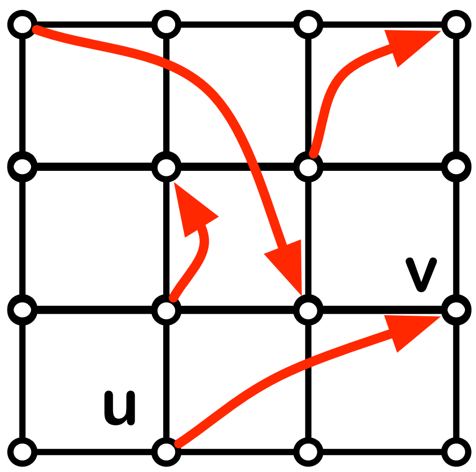

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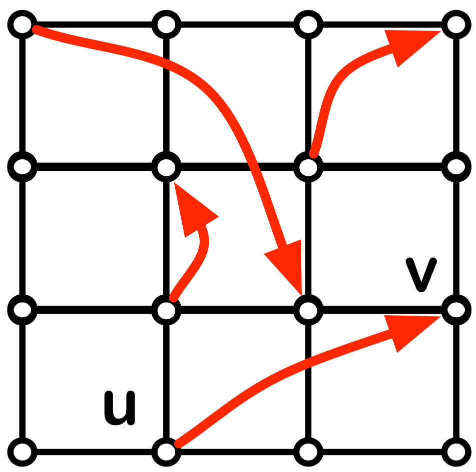

➡ Pr ≥ 1/log n to go twice closer to v.

➡ Greedy routing: $O(\log^2 n)$.

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  - a graph G
  - + 1 random link / node u along some distribution φ_u: $\varphi_u(v)=\Pr\{u \rightarrow v\}$.

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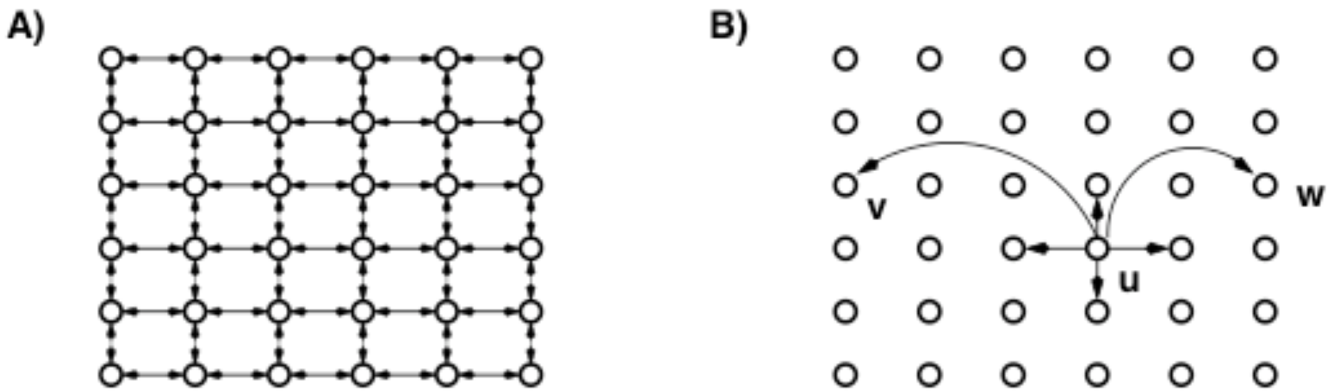
Ex:



$\Pr\{u \rightarrow v\}$ prop. to $1/|u-v|^2$

[Kleinberg 2000]

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- Example:
  - d-dim° meshes are $O(\log^2 n)$-navigable.

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- **Polylog(n)-navigable graphs**:
  - d-dim° meshes [Kleinberg'00]
  - Bounded growth [Duchon, Hanusse,L.,Schabanel'05]
  - Bounded doubling dimension [Slivkins'05]
  - Bounded treewidth [Fraigniaud'05]
  - Excluding a fixed minor [Abraham, Gavoille'06].

Universal augmentation?

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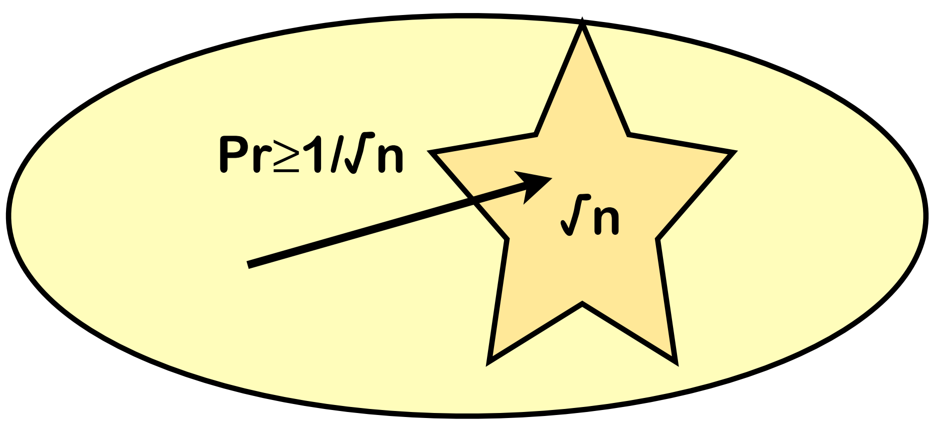

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A) A posteriori universal augmentation :

➡ **All graphs are $\tilde{O}(n^{1/3})$-navigable. [SPAA'07]**

B) A priori universal augmentation :

➡ **Augmentation matrices and (labeling + matrix) -augmentation schemes.**

Universal augmentation?

(A) All graphs are $\tilde{O}(n^{1/3})$-navigable.

Greedy diameter $O(n^{2/5})$

- **Augmentation:**
  - **Pr 1/2** : u picks v uniformly in $B_u(n^{2/5})$.
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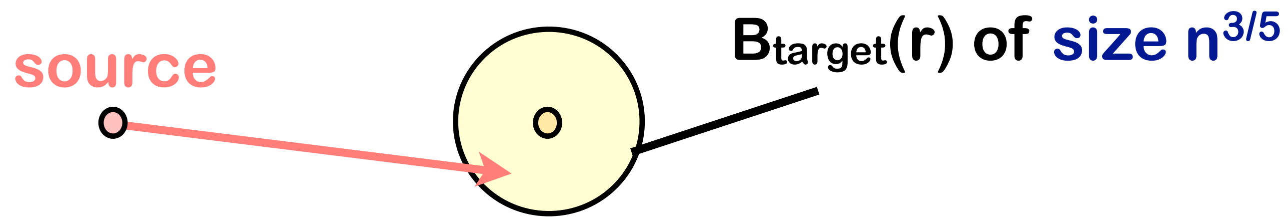

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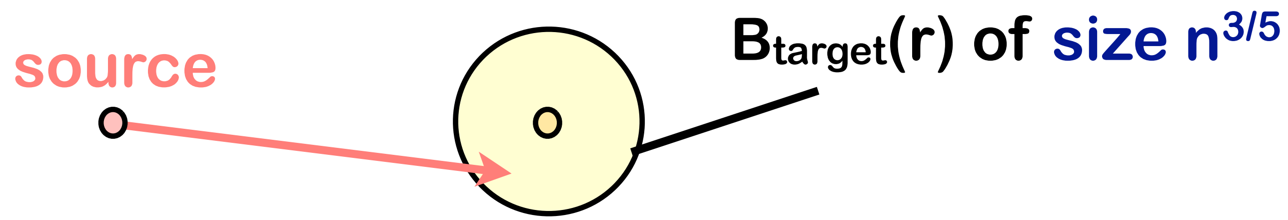

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$= 1/(2n^{2/5})$

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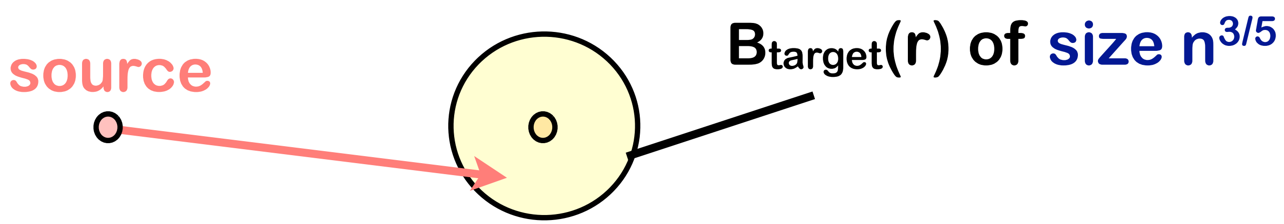

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➡ After $O(n^{2/5})$ steps, greedy routing enters the yellow ball (on expectation).

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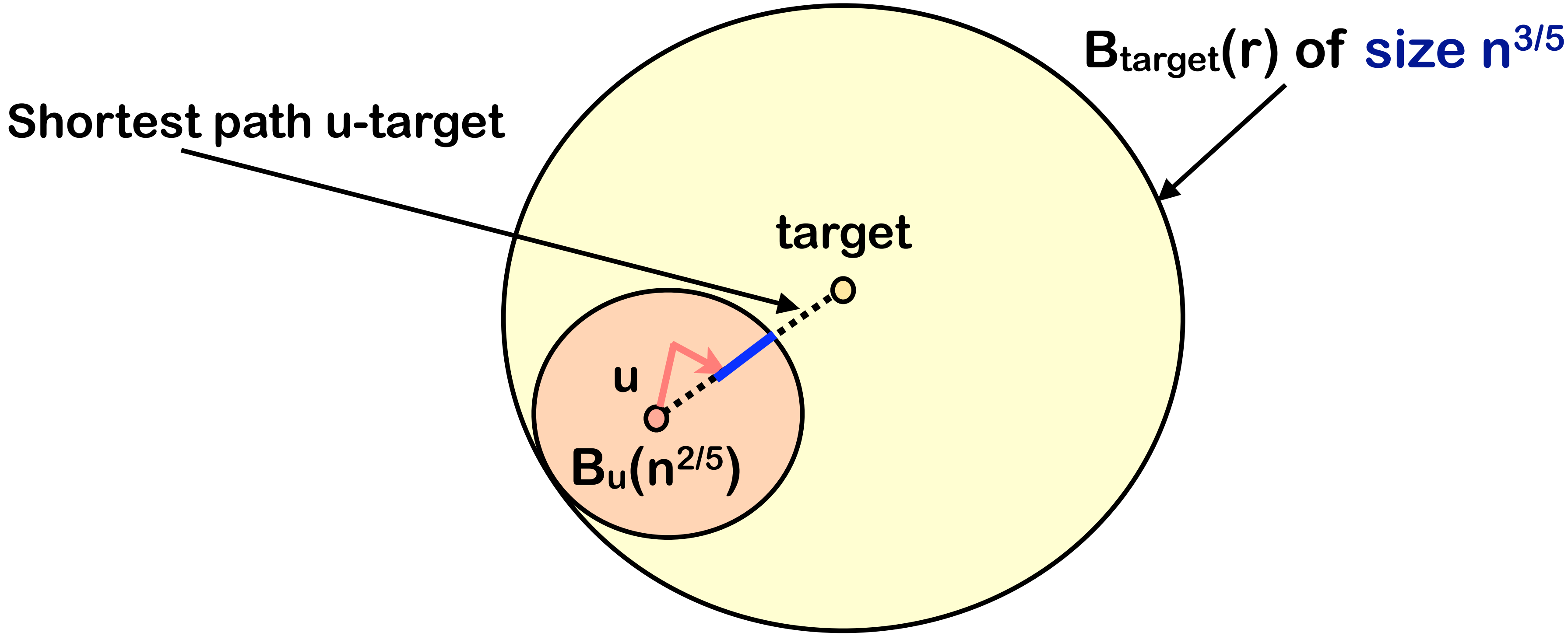

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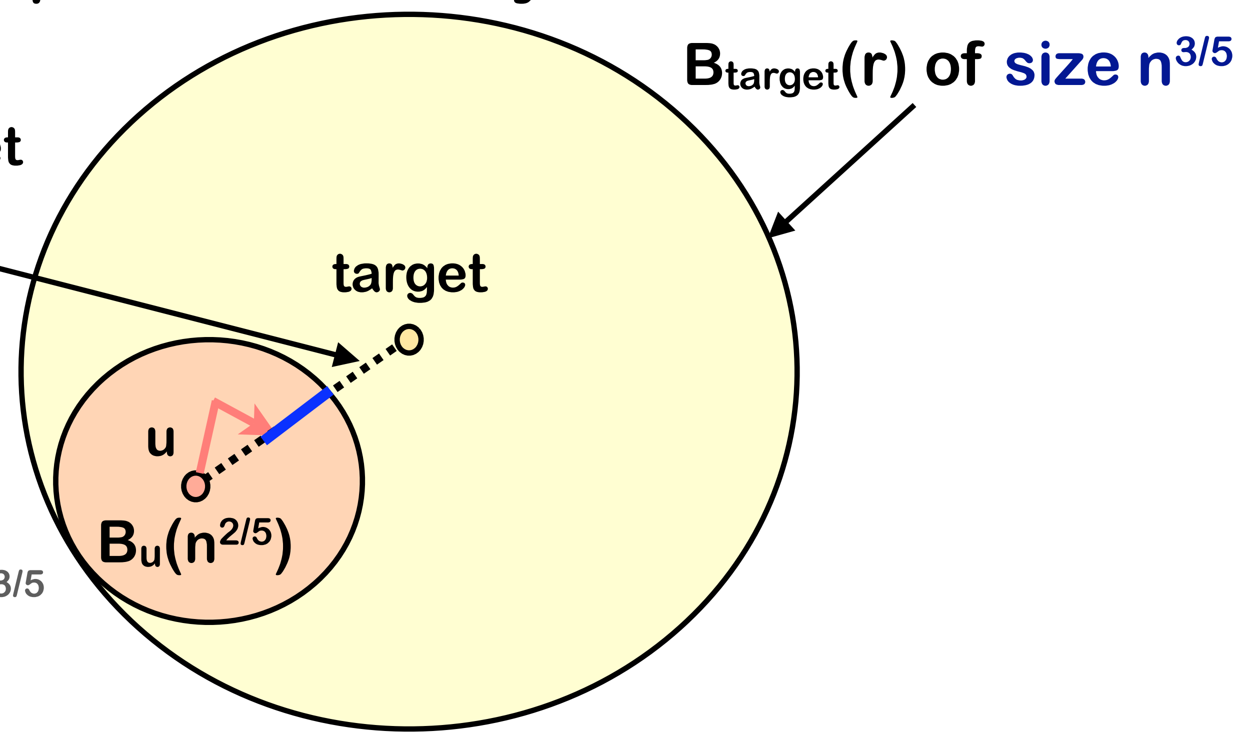

$\underline{Pr(u \rightarrow \text{blue})}$

$\geq$

$(1/2) \times (n^{2/5}/2) \,/\, B_u(n^{2/5})$

$= \Omega\,(1/n^{1/5})$

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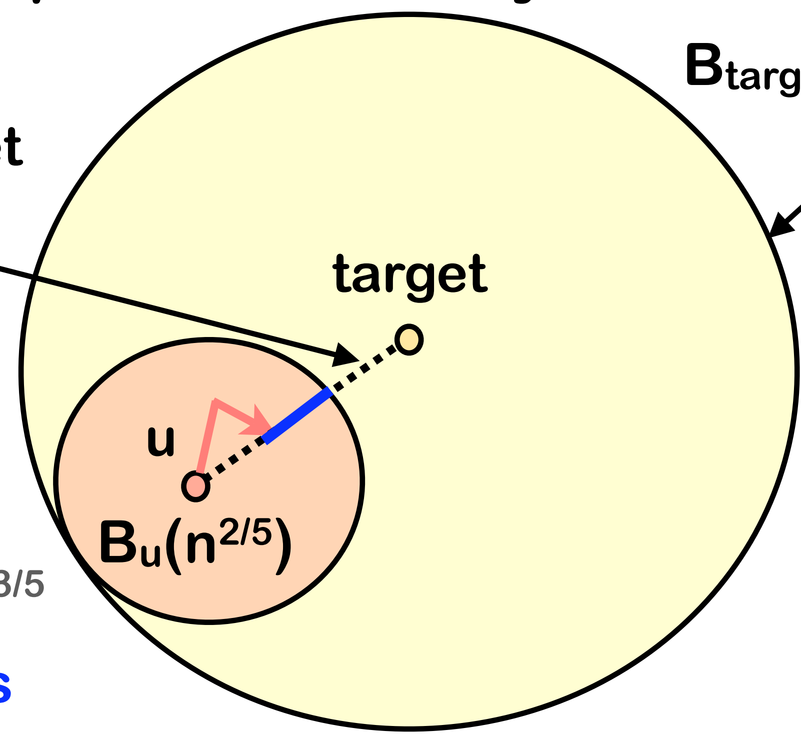

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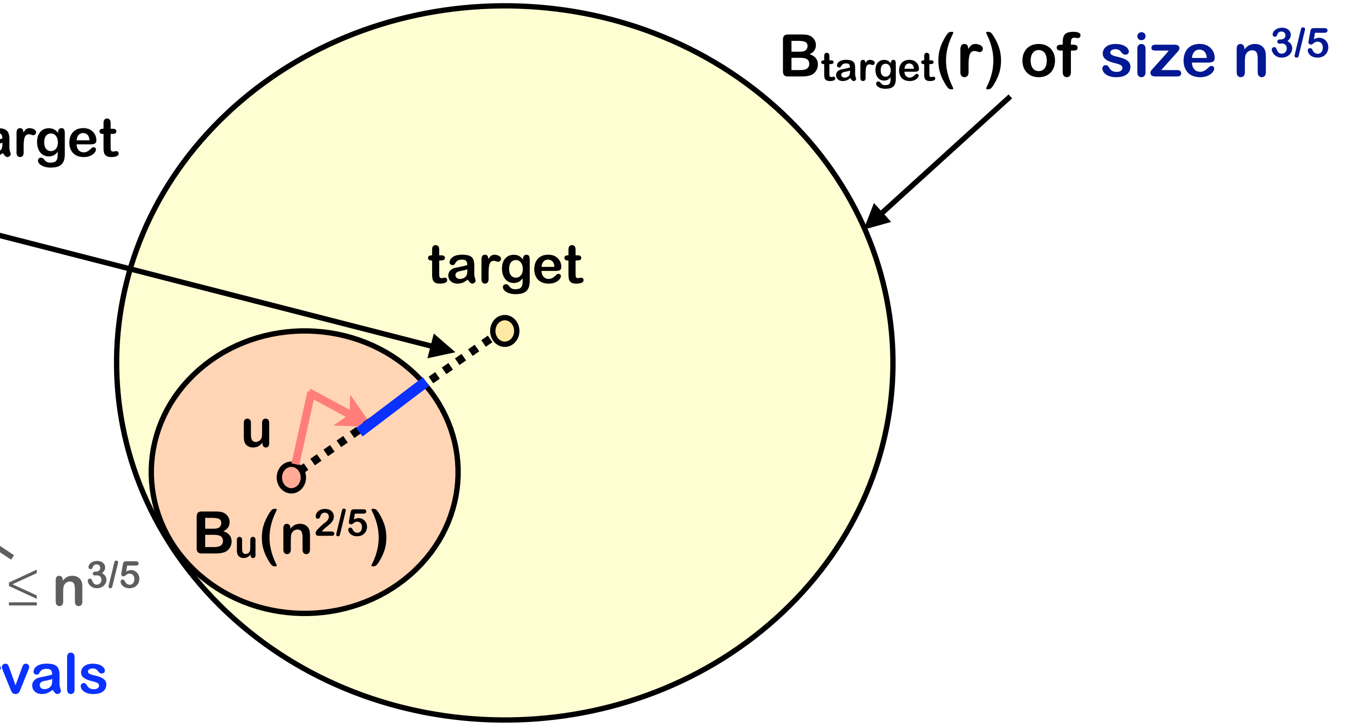

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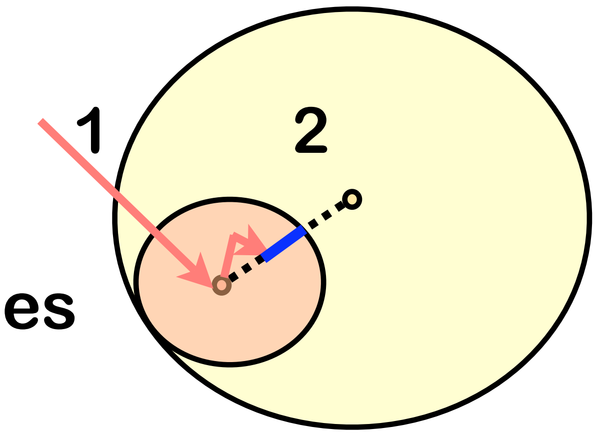
$E(\#\ steps) \leq O(n^{2/5}) + O(n^{1/5}) \times n^{1/5} = O(n^{2/5})$.

Greedy diameter $\tilde{O}(n^{1/3})$

- **Key point for $O(n^{2/5})$:**

  1- Enter a set of reduced size.

  2- Progression inside by picking nodes in smaller sets.



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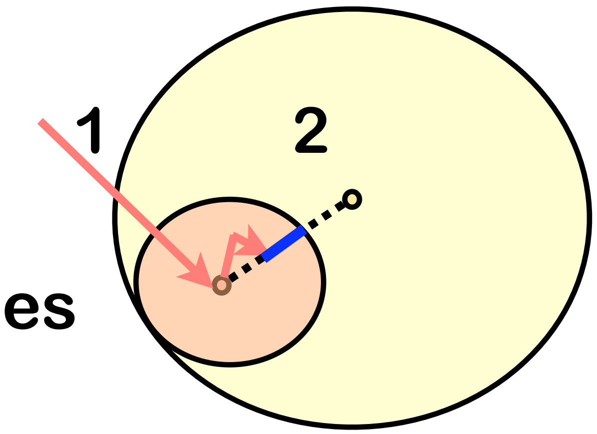


- New augmentation :
  - u picks a level i between 1 and log n.
  - u picks v uniformly in $B_u(2^i)$.
- THM : greedy diameter $\tilde{O}(n^{1/3})$ for any graph.

Universal augmentation?

(B) Augmentation matrices

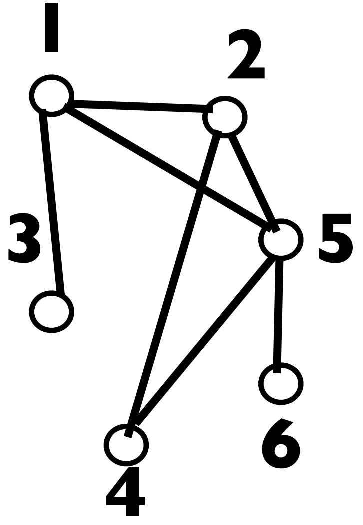
Augmentation matrices

- An augmentation matrix:

Links destinations (columns); nodes (rows)

	1	2	3	4	5	6
1	0	1/2	1/4	1/4	0	0
2	1/3	1/3	1/3	0	0	0
3	1/5	0	1/5	1/5	1/5	1/5
4	1/2	0	0	0	0	1/2
5	0	1/8	1/8	1/8	1/8	1/2
6	1/6	1/3	1/6	1/6	0	1/6

$\rightarrow \Sigma \leq 1$



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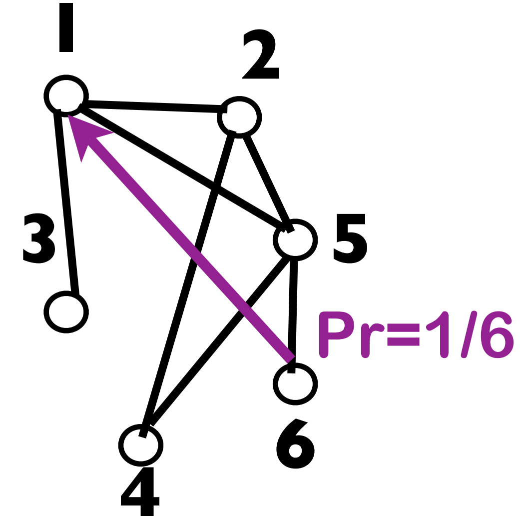

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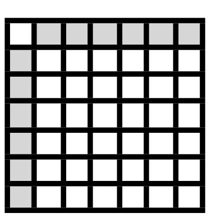

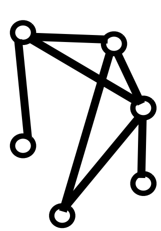

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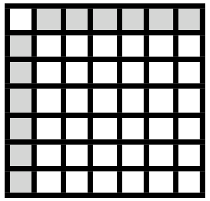

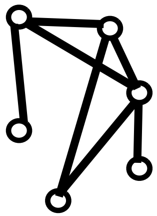

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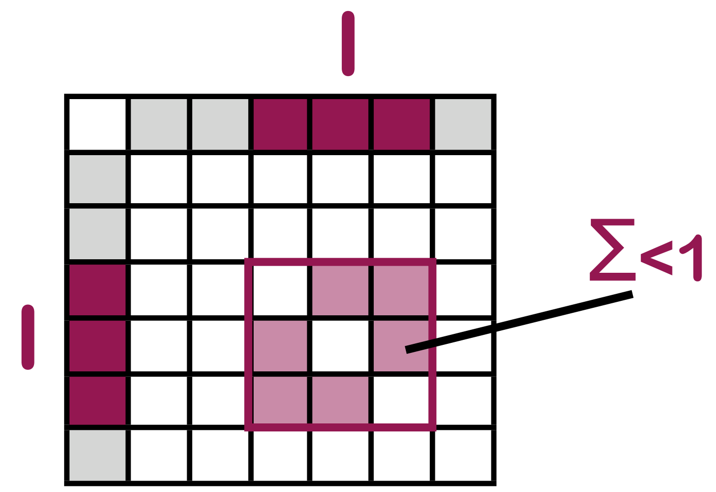
Answer: no.

Anonymous augmentation

- **THM:** For any augmentation matrix, it is possible to label the path to get greedy diameter $\Omega(\sqrt{n})$.

Lemme : for any augmentation matrix M, there is a set **I** of √n indices s.t. $\Sigma p_{ij}<1$ for all i,j in **I**.



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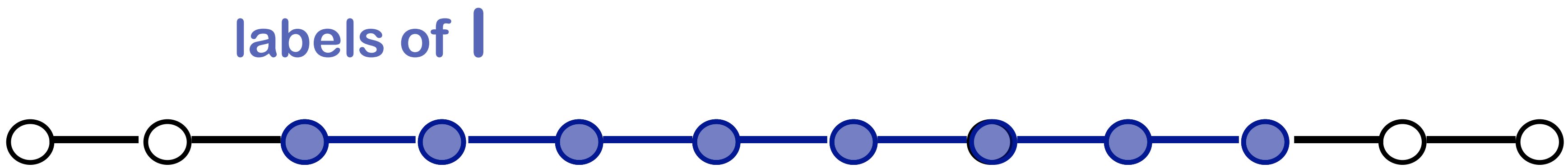

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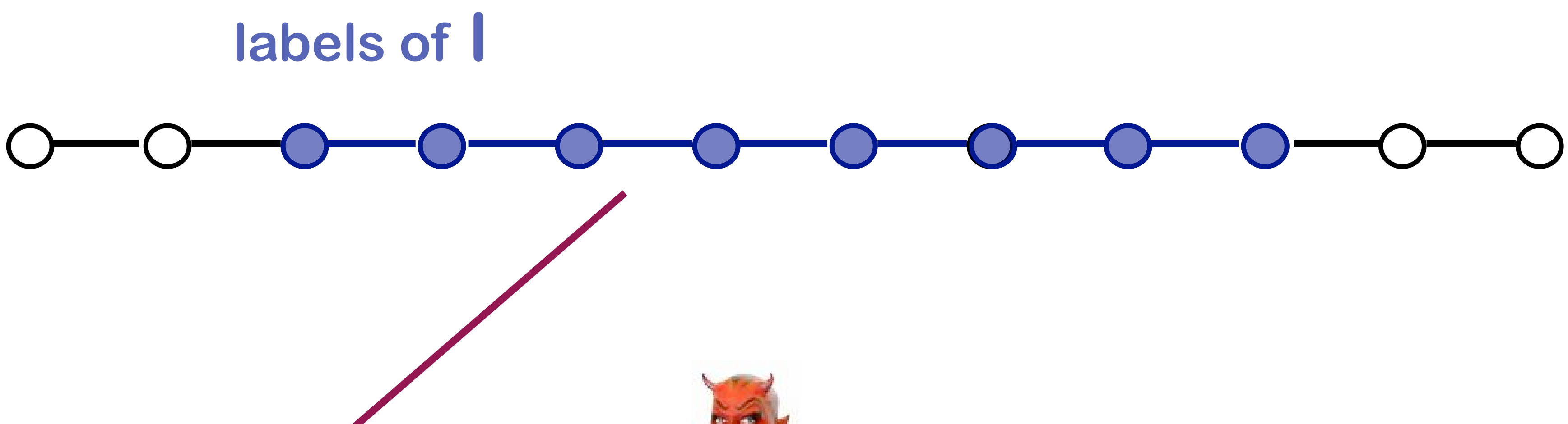

E(# extra links) = $\sum p_{ij} < 1$

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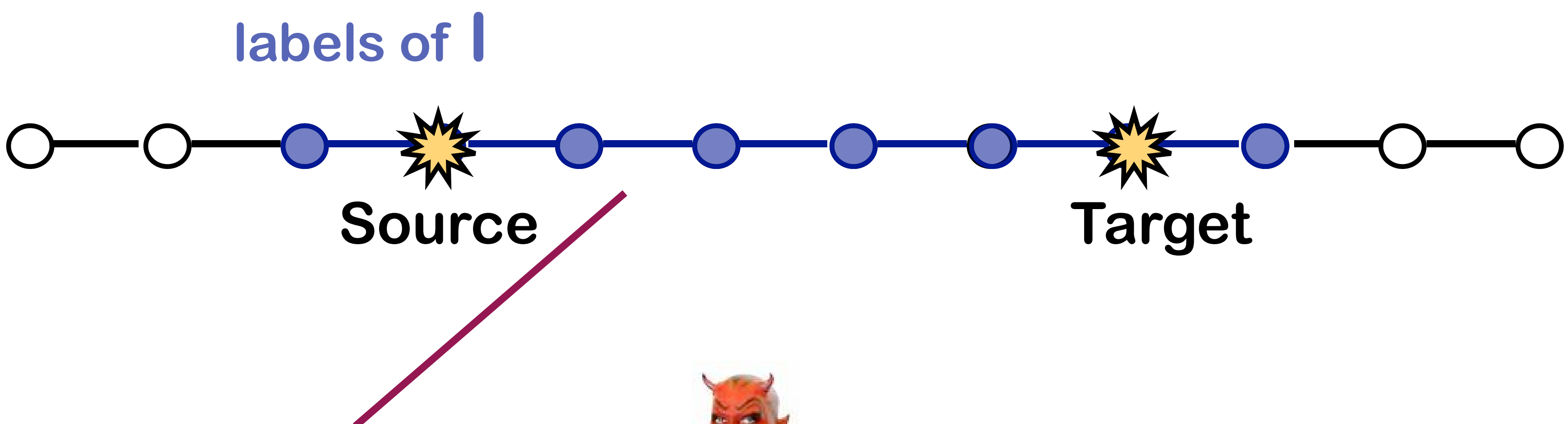

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    Algo(Matrix M, Labeling L)[G]

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➡ Yes for a large graph class.

Matrix + special labeling

- Augmentation matrix of Kleinberg's path:

Matrix + special labeling

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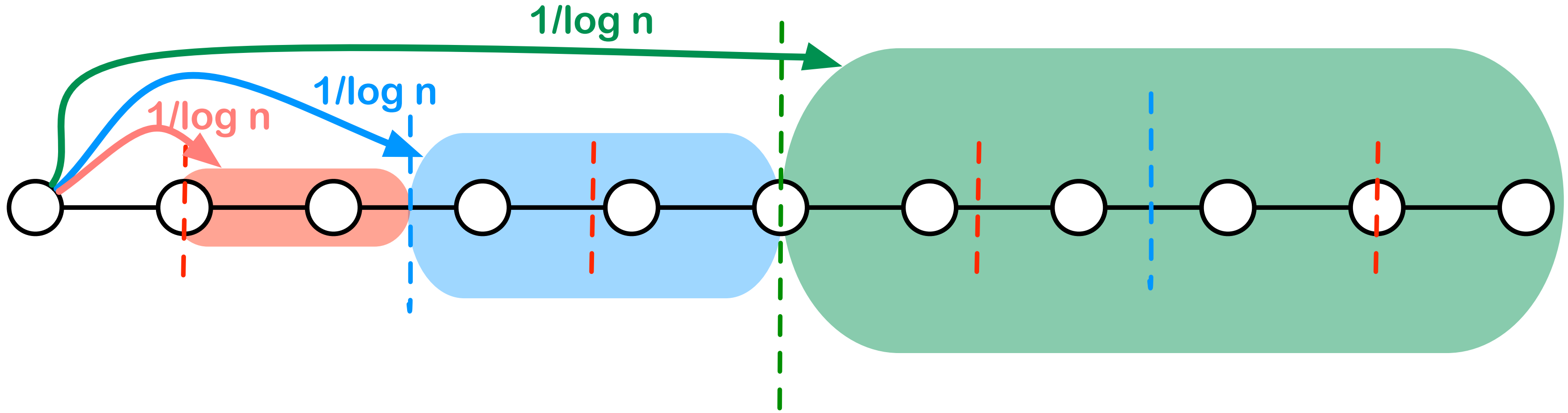

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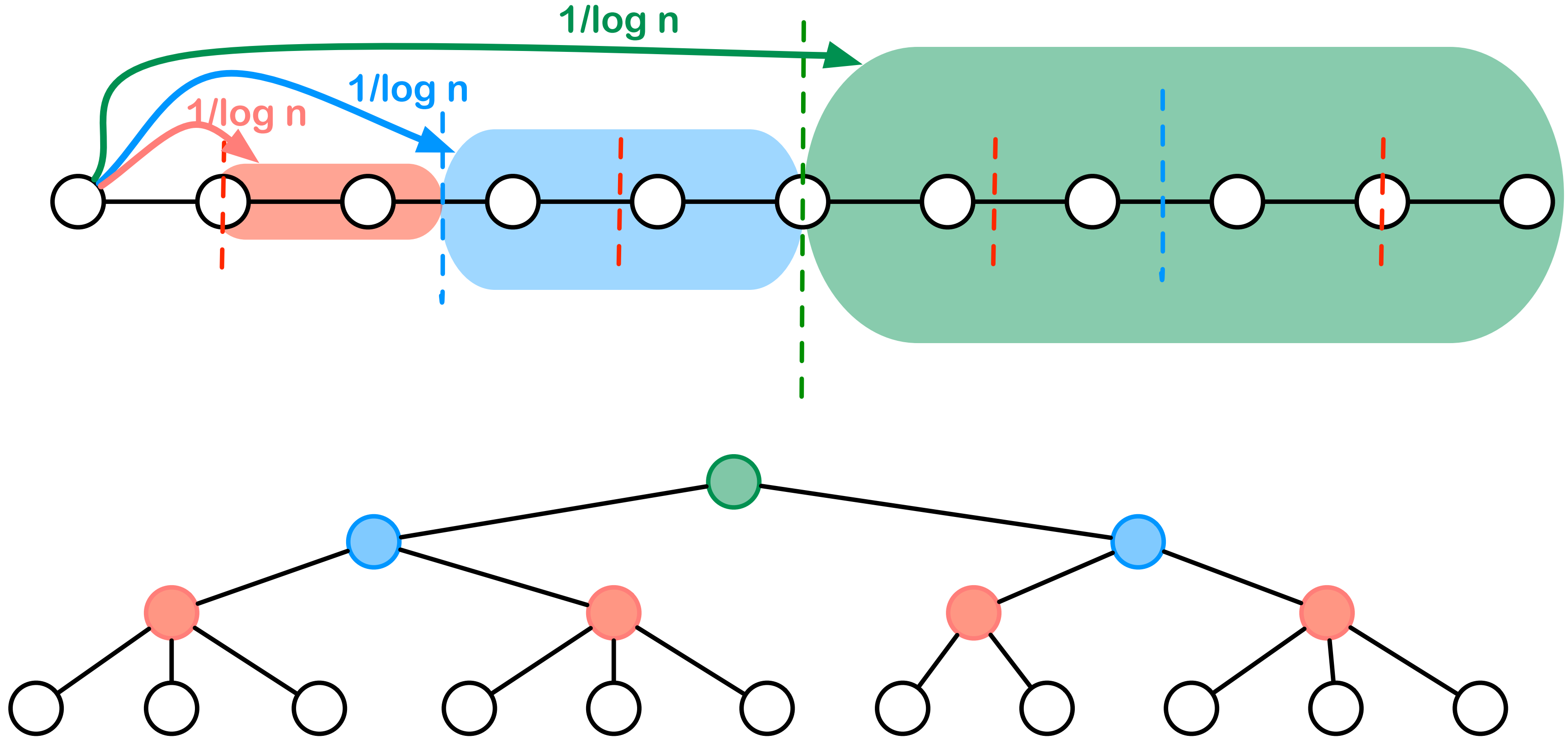

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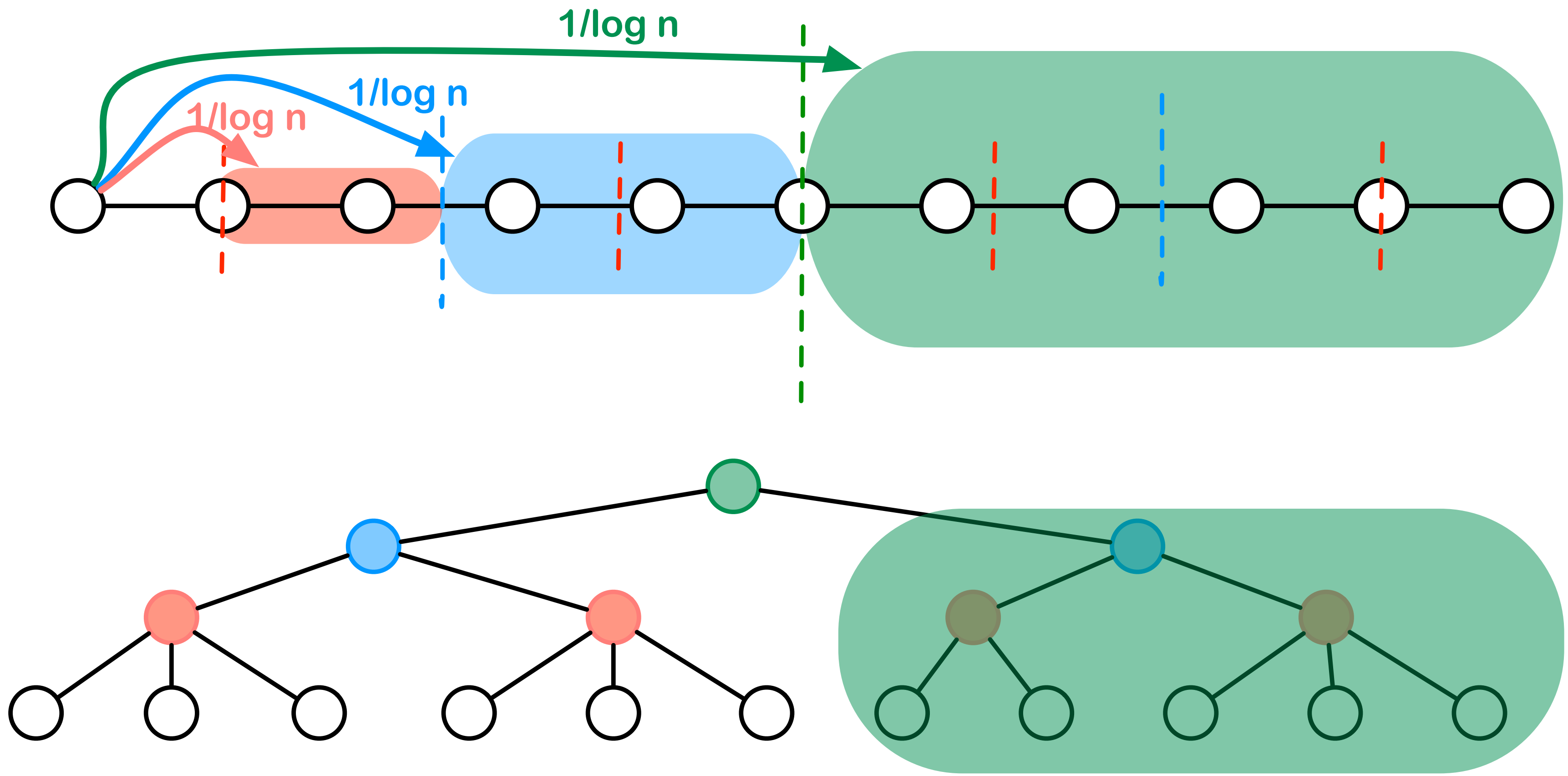

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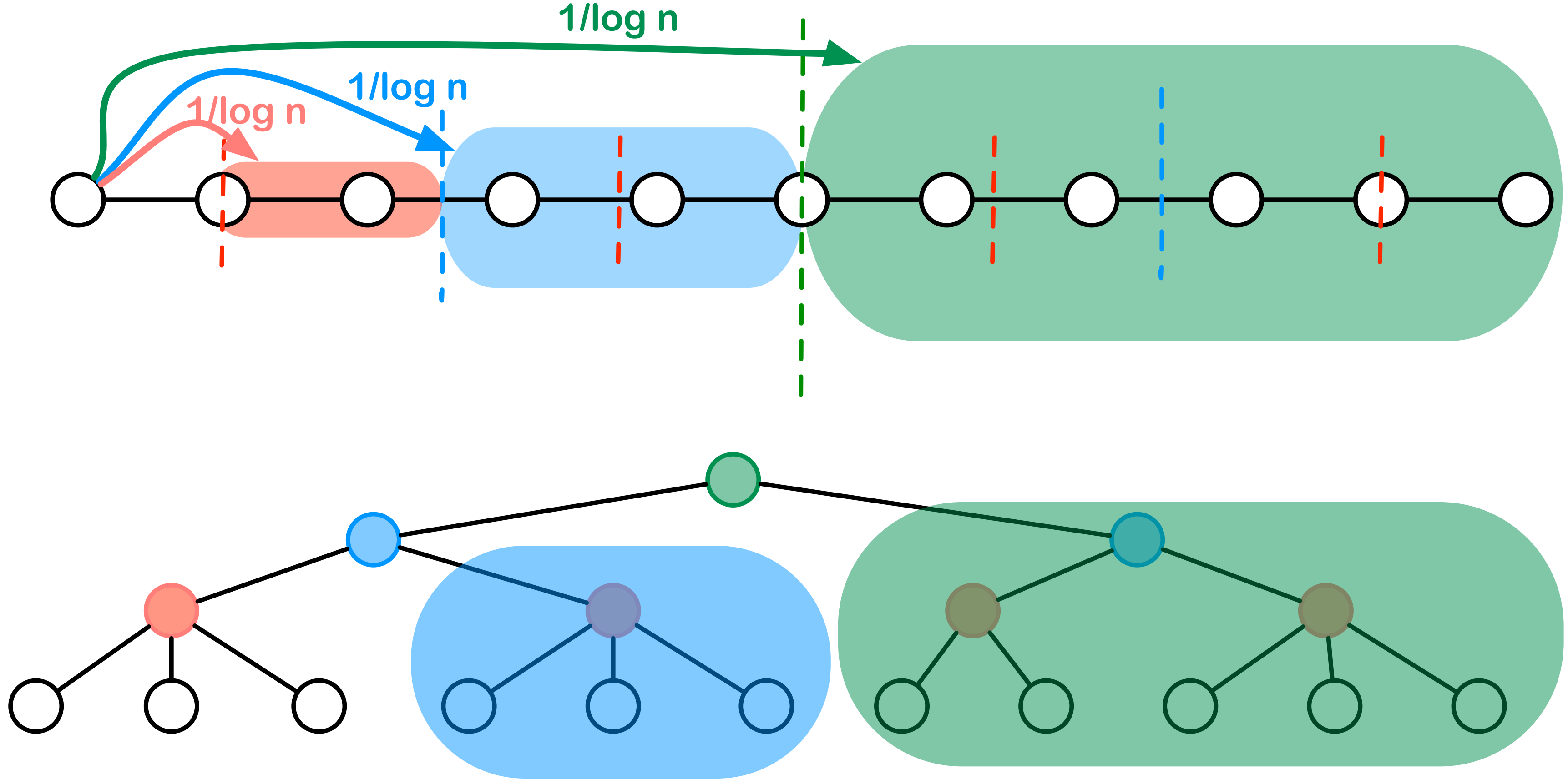

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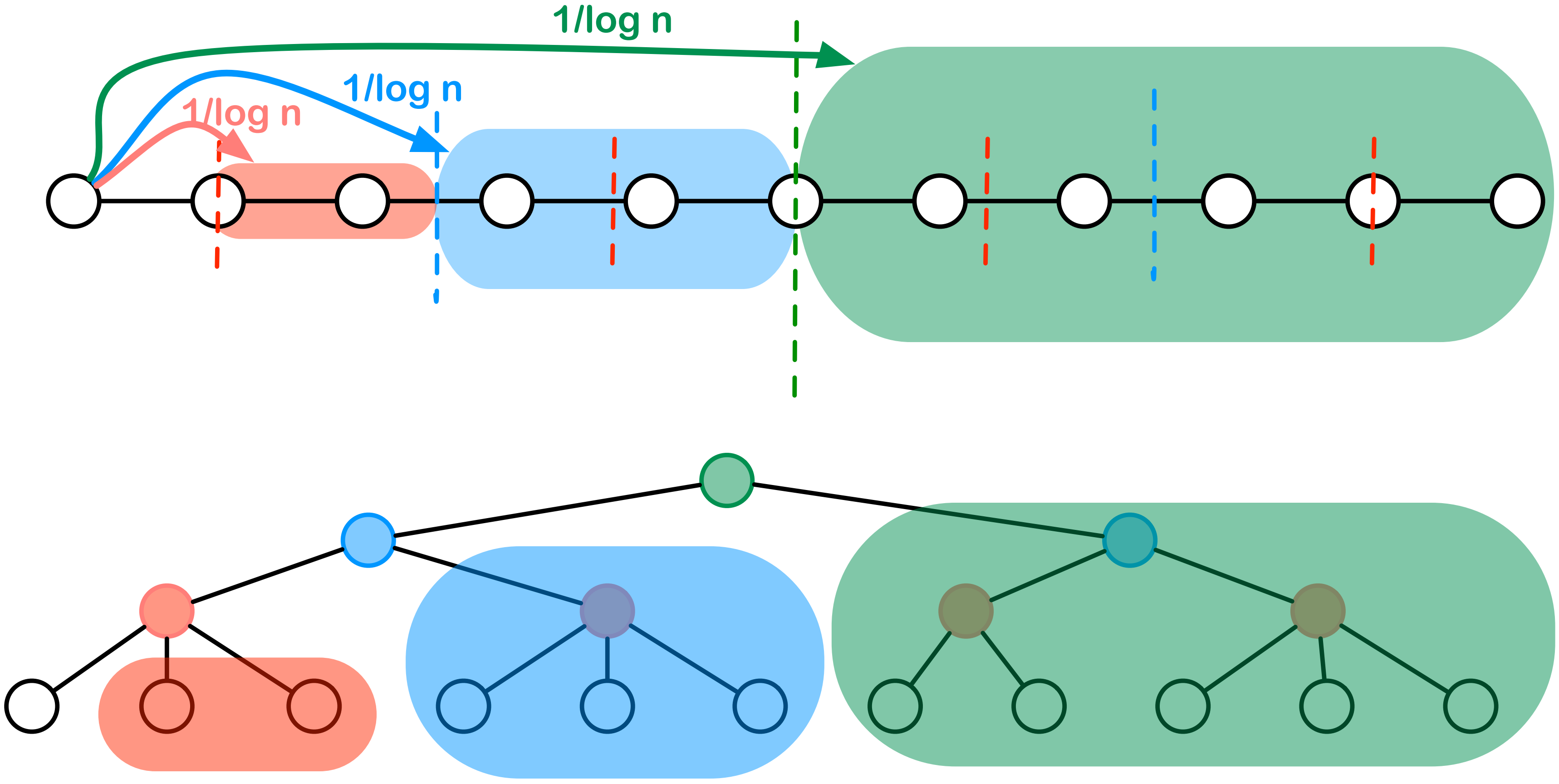

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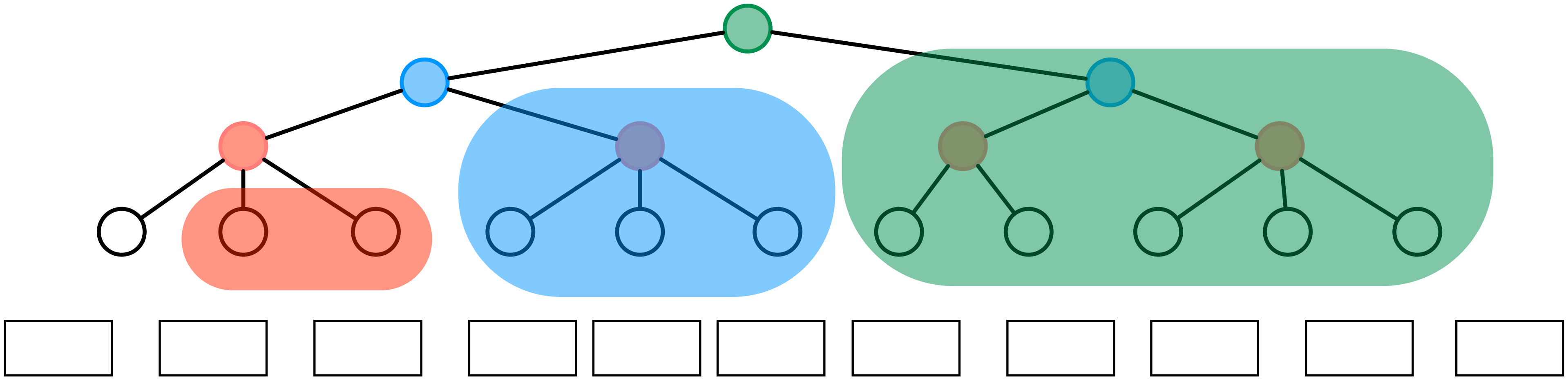

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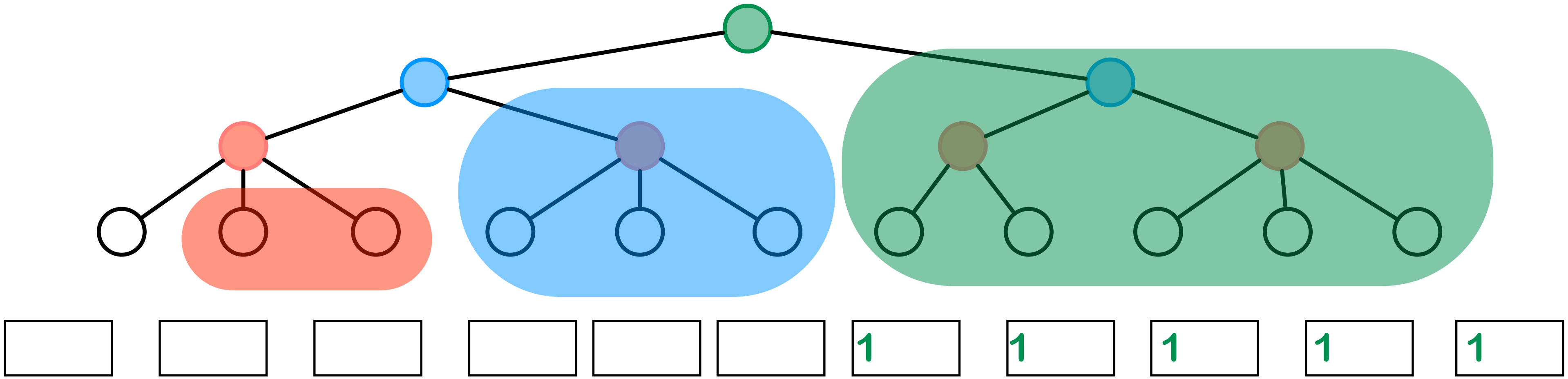

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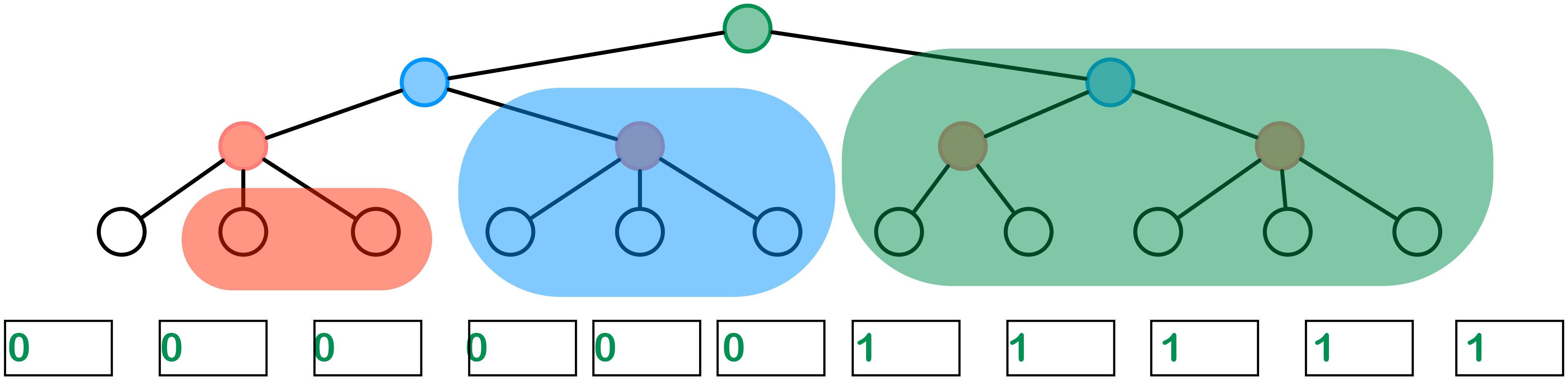

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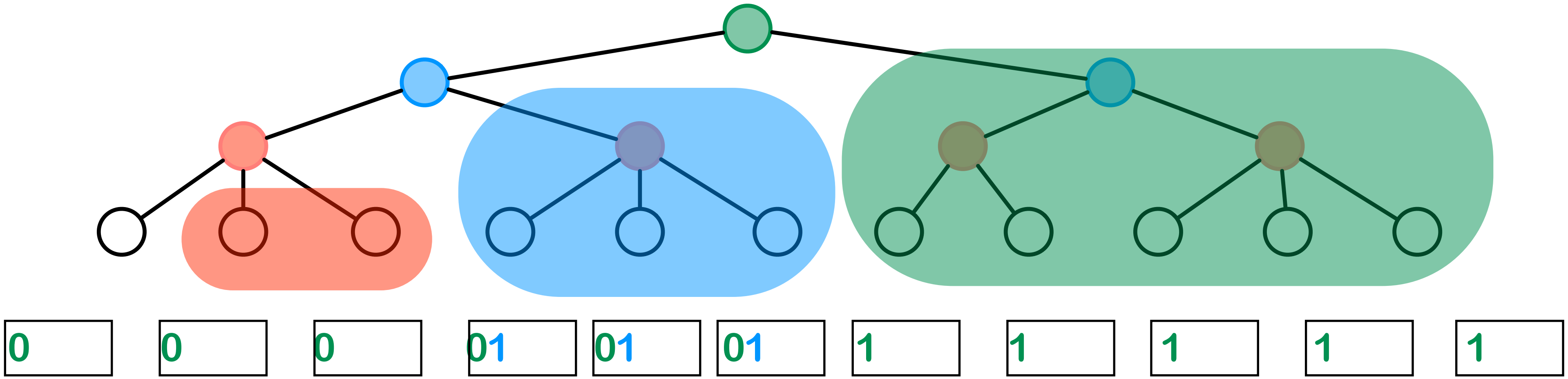

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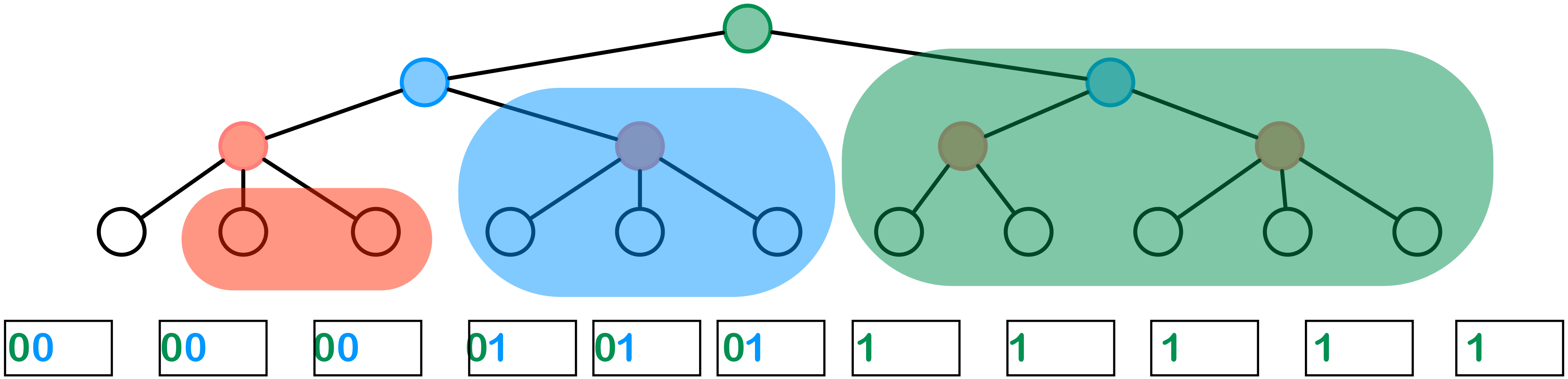

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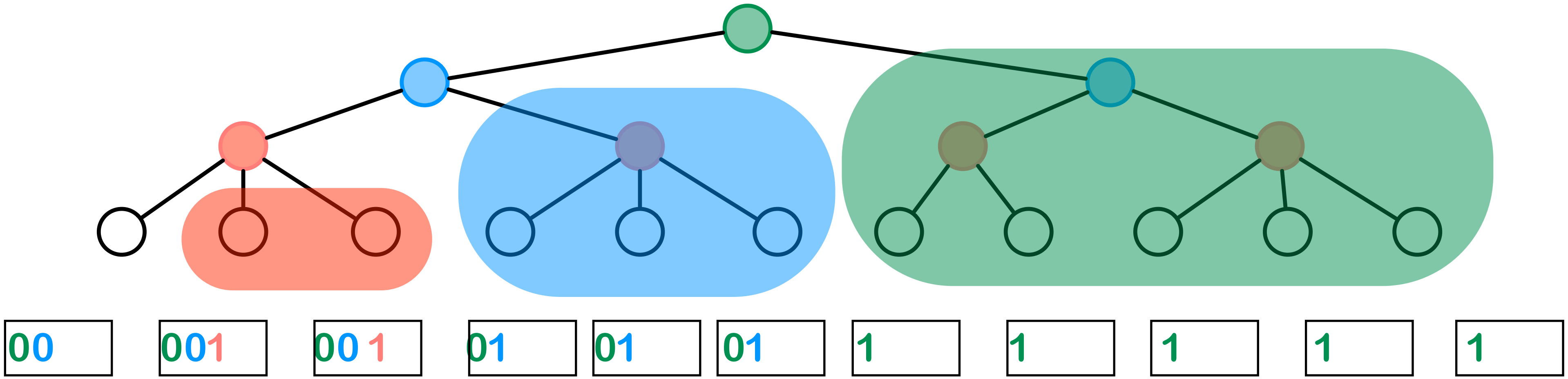

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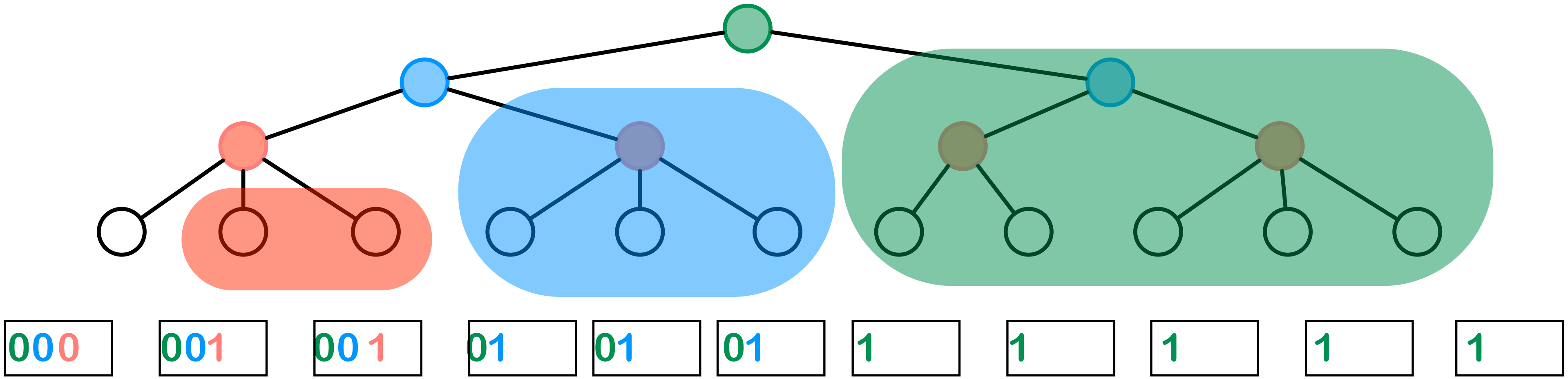

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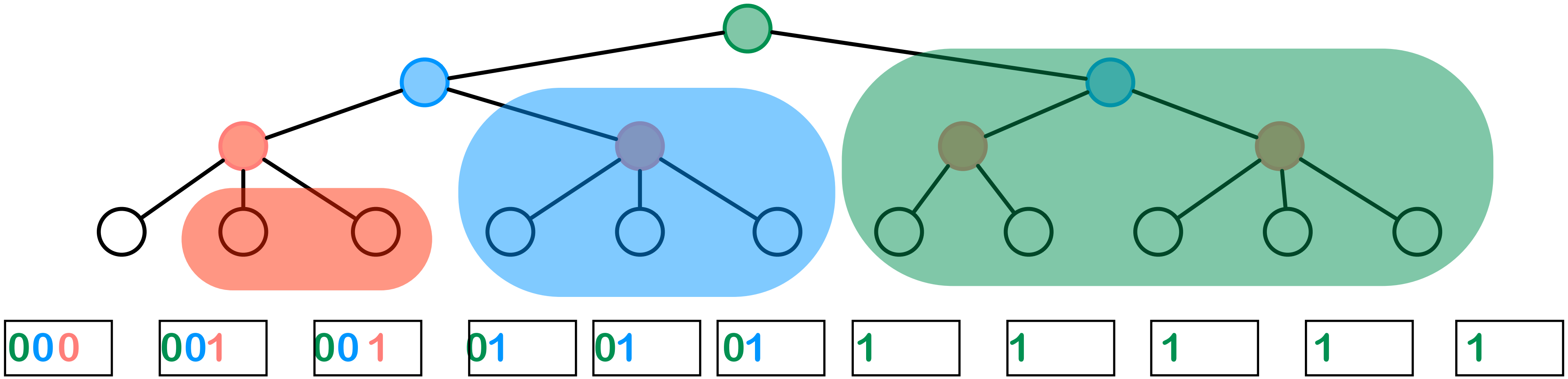

	000	001	001	01	01	01	1
000	0	p	0	p	0	0	p
001	p	0	0	p	0	0	p
001	p	0	0	p	0	0	p
01				0			p
01					0		
01						0	
0							0

...

p = 1/log n

Matrix + special labeling

- The idea:

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- **The idea:**

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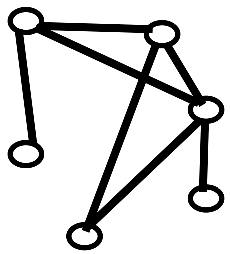

Graph G

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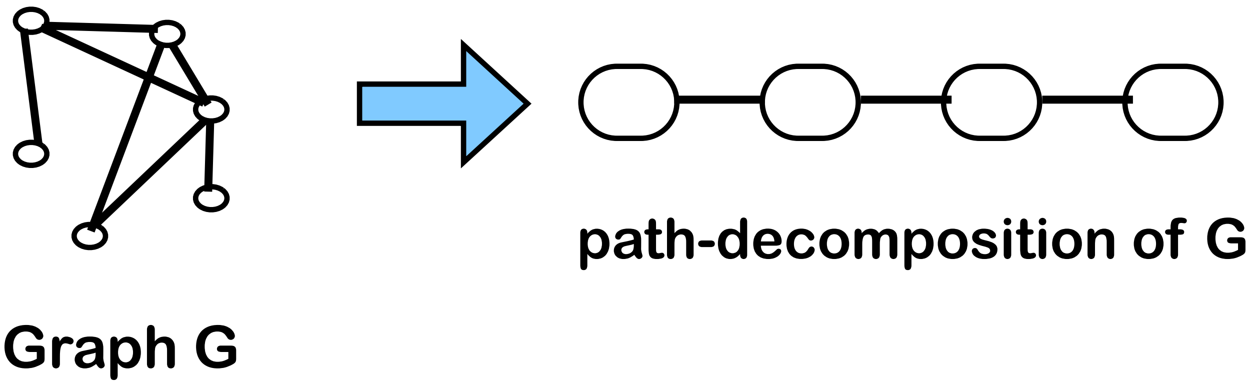

Matrix + special labeling

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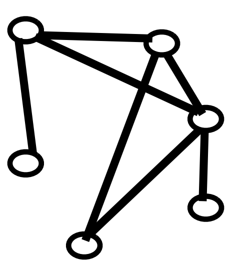
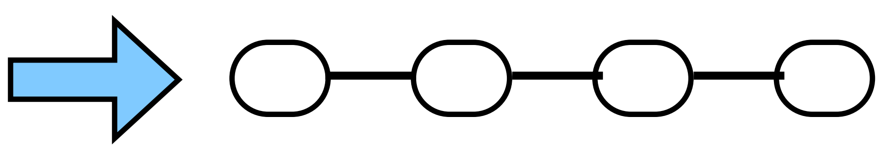


Graph G



path-decomposition of G

labeling

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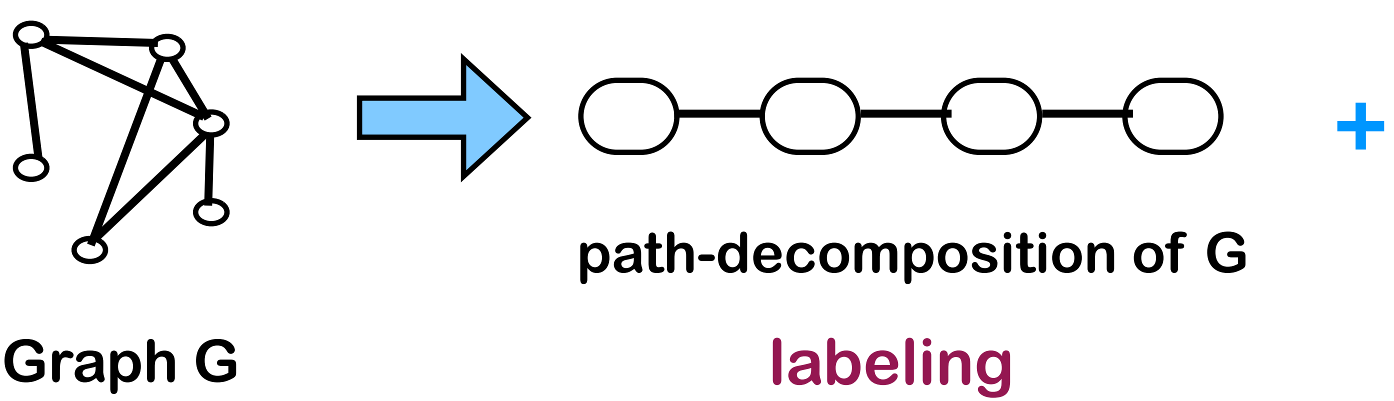

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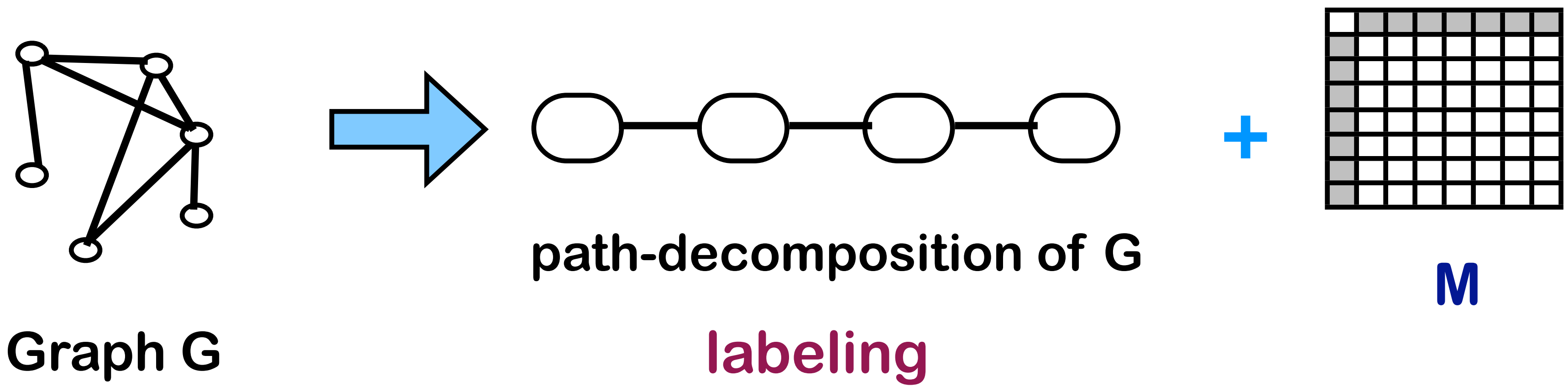
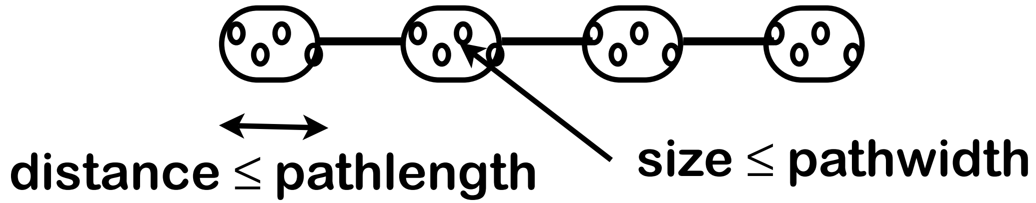


Matrix + special labeling

- <u>Pathshape</u> = min(pathwidth, pathlength). [def°]



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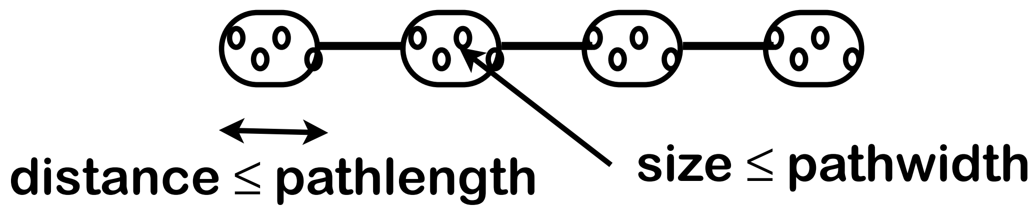

- THM : there is a matrix M and a labeling scheme $\mathcal{E}$ s.t. any G labeled with $\mathcal{E}$ and augmented with M has greedy diameter:

$$O(\ \min(\ \mathrm{pathshape}(G) \times (\log n)^2, \sqrt{n})\ )$$

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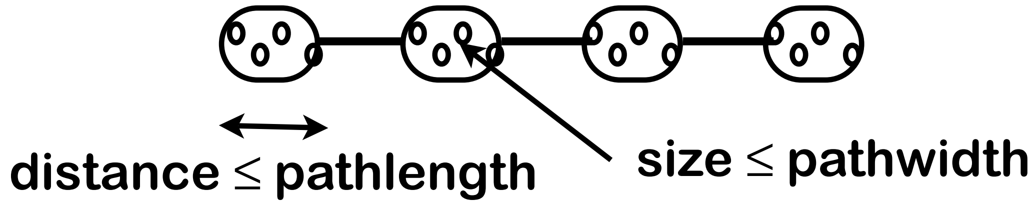

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➡ Trees are $O(\log^3 n)$-navigable. (pathwidth=logn)

➡ AT-free graphs are $O(\log^2 n)$-navigable. (pathlength=C)

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➡ **Is it possible to do better than $O(\sqrt{n})$ with matrices?**

Thank you

- **Ref:**

  Universal Augmentation Scheme for Network Navigability: Overcoming the $\sqrt{n}$-Barrier, *P. Fraigniaud, C. Gavoille, A. Kosowski, Z. Lotker, E. Lebhar, SPAA 2007.*

- **www.liafa.jussieu.fr/~elebhar/SPAA07a.pdf**